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EIGENOSCILLATIONS OF AN ELASTIC BODY WITH A ROUGH SURFACE

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Explicit presentations for the initial terms of the asymptotic solution of the spectral problem of the elasticity theory in a plane region with a rapidly oscillating boundary are obtained. Based on asymptotic formulas, two methods for problem modeling are proposed: with the use of Wenzel's boundary conditions and with the use of the principle of a smooth image of a singularly perturbed boundary. Various approaches to justification of asymptotic presentations are discussed.

Key words: *anisotropic elastic body, fine-grain boundary, asymptotic presentations.*

1. Body with a Rough Surface. Let Ω be a plane inhomogeneous anisotropic elastic body bounded by a simple smooth closed contour Γ. The contour length is reduced to a unit length by scaling. Using a small parameter $h = N^{-1}$, where N is a large natural number, we determine a rapidly oscillating (fine-grain) boundary of the body Ω_h (Fig. 1):

$$\Gamma_h = \{x \in O_\Gamma \colon n = hH(s, h^{-1}s)\}. \tag{1.1}$$

Here O_Γ is the neighborhood of the set Γ on which an orthogonal system (n, s) of curvilinear coordinates is introduced, s is the arc length on Γ, n is the oriented distance to the contour Γ ($n > 0$ outside Ω), and H is a smooth function of the slow s and fast $\eta_2 = h^{-1}s$ variables, which is periodic with respect to the latter variable (in the present work, a unit period is used).

We consider the problem of eigenoscillations of the body Ω_h:

$$-\partial_{x_1}\sigma_{1k}(u; h, x) - \partial_{x_2}\sigma_{2k}(u; h, x) = \Lambda(h)\rho(x)u_k(h, x), \qquad x \in \Omega_h; \tag{1.2}$$

$$\sigma_k^{(\nu)}(u; h, x) := \nu_1(h, x)\sigma_{1k}(u; h, x) + \nu_2(h, x)\sigma_{2k}(u; h, x) = 0, \qquad x \in \Gamma_h. \tag{1.3}$$

Here $\rho > 0$ and $\Lambda(h) \geq 0$ are the material density and the squared eigenfrequency, $\partial_{x_j} = \partial/\partial x_j$, $\nu = (\nu_1, \nu_2)$ is the unit vector of the external normal to the boundary $\partial\Omega_h = \Gamma_h$, $u = (u_1, u_2)$ is the vector of displacements, and σ_{jk} are the Cartesian components of the stress tensor:

$$\sigma_{jk}(u; h, x) = \sum_{p,q=1}^{2} A_{jk}^{pq}(x)\varepsilon_{pq}(u; h, x), \qquad \varepsilon_{pq}(u) = \frac{1}{2}\Big(\frac{\partial u_p}{\partial x_q} + \frac{\partial u_q}{\partial x_p}\Big).$$

As the domain Ω_h for $H > 0$ is wider than the domain Ω, the components A_{jk}^{pq} of the quadrivalent tensor A, which is smooth, symmetric, and positively defined, we assign the sets $\overline{\Omega} = \Omega \cup \partial\Omega$ in the neighborhood $\Omega \cup O_\Gamma$. Other fields, which are independent of the parameter h, are also assumed to be smoothly continued into the external domain with respect to Ω. For brevity, we do not indicate the parameter h among the function arguments in what follows. In further notation, we do not distinguish between the contour point Γ and its coordinate s.

Sections 2–4 describe an asymptotic analysis of the spectral problem formulated above, and essentially explicit formulas for two terms of the asymptotic expansions of its solutions are derived. The main challenge of

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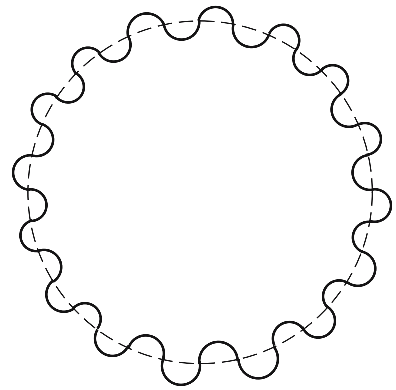

Fig. 1. Body with a rough surface.

the paper, however, is modeling the problem in the domain with a rapidly oscillating boundary (1.1), namely, constructing simpler formulated boundary-value problems whose solutions yield an approximation with elevated accuracy (two-term asymptotics) for the solution $\{\Lambda, u\}$ in the domain Ω_h. For the scalar Dirichlet problem, various modeling methods were developed in [1]. In addition to technical problems associated with definition of the terms of the asymptotic ansatz, the problem of the elasticity theory with boundary conditions in stresses (analog of the Neumann conditions) involve one more difficulty: the spectral parameter is present in the boundary conditions of the limiting and resulting problems.

2. Constructing the Main Terms of the Asymptotics. It is known that a boundary layer appears near a rapidly oscillating boundary (see, e.g., [2, 3]); therefore, the asymptotic ansatz for the solution $\{\Lambda, u\}$ of the spectral problem (1.2), (1.3) is taken in the form

$$\Lambda = \lambda_0 + h\lambda_1 + \dots; \tag{2.1}$$

$$u(x) = v^0(x) + hv^1(x) + \chi(n)h(w^1(s,\eta) + hw^2(s,\eta)) + \dots\ . \tag{2.2}$$

Here v^i are terms of the regular type and w^i are terms of the boundary-layer type multiplied by a patch function χ equal to unity near the contour Γ and to zero outside the set O_Γ. Terms of the regular type are solutions of problems in the domain $\Omega = \Omega_0$ bounded by the limiting contour $\Gamma = \Gamma_0$; for instance, the formal transition to $h = 0$ leads to relations

$$-\partial_{x_1}\sigma_{1k}(v^0; x) - \partial_{x_2}\sigma_{2k}(v^0; x) = \lambda_0\rho(x)v_k^0(x), \qquad x \in \Omega; \tag{2.3}$$

$$\sigma_k^{(n)}(v^0; x) := n_1(s)\sigma_{1k}(v^0; x) + n_2(s)\sigma_{2k}(v^0; x) = 0, \qquad x \in \Gamma. \tag{2.4}$$

As the normal $n = (n_1, n_2)$ to the contour Γ differs from the oscillating normal ν to the contour Γ_h, the solution of problem (2.3), (2.4) leaves a residual in the boundary condition (1.3), which is compensated by terms of the boundary-layer type. The vector functions w^i depend not only on the slow variable $s \in \Gamma$ but also on the fast variables

$$\eta = (\eta_1, \eta_2) = (h^{-1}n, h^{-1}s). \tag{2.5}$$

Substitution of coordinates $x \mapsto \eta$ and the local-periodic structure of the boundary (1.1) are responsible for the emergence of a half-band

$$\Pi(s) = \{\eta \in \mathbb{R}^2\colon \eta_2(0,1), \eta_1 < H(s,\eta_2)\} \tag{2.6}$$

as another limiting domain; the butt-end face of the half-band $\pi(s) = \{\eta \in \partial\Pi(s)\colon\ \eta_1 \in (0,1)\}$ is curved. We introduce the projections of the vector w^i onto the axes n and s:

$$\boldsymbol{w}_1^i = w_n^i = n_1w_1^i + n_2w_2^i, \qquad \boldsymbol{w}_2^i = w_s^i = -n_2w_1^i + n_1w_2^i.$$

As the stretching of coordinates by a factor of h^{-1} and the transition to $h = 0$ implies “freezing” of dependences on slow variables at the points $s \in \Gamma$, the coordinates (2.5) should be considered mainly as Cartesian coordinates. We assume that $\boldsymbol{w}^i = (\boldsymbol{w}_1^i, \boldsymbol{w}_2^i)$ and

$$\boldsymbol{\sigma}_{jk}(\boldsymbol{w}; s, \eta) = \sum_{p,q=1}^{2} \boldsymbol{A}_{jk}^{pq}(0, s) \boldsymbol{\varepsilon}_{pq}(\boldsymbol{w}; s, \eta), \qquad \boldsymbol{\varepsilon}_{pq}(\boldsymbol{w}) = \frac{1}{2}\Big(\frac{\partial \boldsymbol{w}_p}{\partial \eta_q} + \frac{\partial \boldsymbol{w}_q}{\partial \eta_p}\Big), \tag{2.7}$$

where $\boldsymbol{A}_{jk}^{pq}(0, s)$ are the components of the tensor $\boldsymbol{A}(0, s)$ obtained by orthogonal transformation of the tensor $A(x)\Big|_{n=0}$ by the matrix

$$\Theta(s) = \left(\begin{array}{cc} n_1(s) & n_2(s) \\ -n_2(s) & n_1(s) \end{array} \right).$$

Preserving the notation $v^i(n, s)$ for the fields written on the set $\Omega \cap O_\Gamma$ in curvilinear coordinates, we obtain $\boldsymbol{v}^i(0, s) = \Theta(s) v^i(0, s)$. We also give formulas for strains and the equation of equilibrium in the coordinate system (n, s):

$$\begin{gathered} \varepsilon_{nn}(u) = \partial_n u_n, \qquad \varepsilon_{ss}(u) = J^{-1}(\partial_s u_s + \varkappa u_n), \\ \varepsilon_{ns}(u) = \varepsilon_{sn}(u) = (\partial_n u_s + J^{-1}(\partial_s u_n - \varkappa u_s))/2; \end{gathered} \tag{2.8}$$

$$\begin{gathered} -\partial_n \sigma_{nn}(u) - J^{-1}(\partial_s \sigma_{ns}(u) + \varkappa(\sigma_{nn}(u) - \sigma_{ss}(u))) = \Lambda \rho u_n, \\ -\partial_n \sigma_{sn}(u) - J^{-1}(\partial_s \sigma_{ss}(u) + 2\varkappa \sigma_{ns}(u)) = \Lambda \rho u_s. \end{gathered} \tag{2.9}$$

Here $J(n, s) = 1 + n\varkappa(s)$ is the Jacobian and $\varkappa(s)$ is the curvature of the arc Γ at the point s. Finally, the projections of the normal ν onto the axes n and s admit the presentations

$$\begin{gathered} \nu_n(s, \eta) = \boldsymbol{\nu}_1(s, \eta_2)(1 - hY(s, \eta)) + O(h^2), \\ \nu_s(s, \eta) = \boldsymbol{\nu}_2(s, \eta_2)(1 - hY(s, \eta)) - h(\boldsymbol{\nu}_2(s, \eta_2)\eta_1 \varkappa(s) + y(s, \eta_2)\partial_s H(s, \eta_2)) + O(h^2), \\ Y = \boldsymbol{\nu}_1 \boldsymbol{\nu}_2(\partial_s H - \varkappa H \partial_{\eta_2} H), \qquad y = (1 + |\partial_{\eta_2} H|^2)^{-1/2}, \end{gathered} \tag{2.10}$$

where $\boldsymbol{\nu} = (\boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$ is the unit vector of the normal to the butt-end face of the half-band $\Pi(s) \subset \mathbb{R}^2 \ni \eta$.

In relations (2.8) and (2.9), we pass to the stretched coordinates (2.5) and substitute ansatz (2.2) into Eqs. (1.2) and boundary conditions (1.3). Collecting coefficients at identical powers of the small parameter and equating their sums to zero, we obtain the following problems in the half-band (2.6):

$$\begin{gathered} -\partial_{\eta_1} \boldsymbol{\sigma}_{1k}(\boldsymbol{w}^i; s, \eta) - \partial_{\eta_2} \boldsymbol{\sigma}_{2k}(\boldsymbol{w}^i; s, \eta) = \boldsymbol{F}_k^i(s, \eta), \qquad \eta \in \Pi(s), \\ \boldsymbol{\sigma}_k^{(\nu)}(\boldsymbol{w}^i; s, \eta) := \boldsymbol{\nu}_1(s, \eta_2)\boldsymbol{\sigma}_{1k}(\boldsymbol{w}^i; s, \eta) + \boldsymbol{\nu}_2(s, \eta_2)\boldsymbol{\sigma}_{2k}(\boldsymbol{w}^i; s, \eta) = \boldsymbol{G}_k^i(s), \quad \eta \in \pi(s), \\ \boldsymbol{w}^i(s, \eta_1, 0) = \boldsymbol{w}^i(s, \eta_1, 1), \qquad \partial_{\eta_2} \boldsymbol{w}^i(s, \eta_1, 0) = \partial_{\eta_2} \boldsymbol{w}^i(s, \eta_1, 1), \qquad \eta_1 < H(s, 0). \end{gathered} \tag{2.11}$$

According to expansion (2.10), we have

$$\boldsymbol{F}_k^0(s, \eta) = 0, \qquad \boldsymbol{G}_1^0(s, \eta) = 0, \qquad \boldsymbol{G}_2^0(s, \eta) = -\boldsymbol{\nu}_2(s, \eta_2)\sigma_{ss}(v^0; 0, s). \tag{2.12}$$

Here $\boldsymbol{G}_2^0$ is the main part of the residual of the vector function v^0 in the boundary condition (1.3): all stresses, except for $\sigma_{ss}(v^0; 0, s)$, are canceled on Γ by virtue of relations (2.4).

The mean value of the component $\boldsymbol{\nu}_2$ over the butt-end face $\pi(s)$ equals zero; hence, problem (2.11) with $i = 0$ has the only solution exponentially decaying at infinity:

$$\boldsymbol{w}^0(s, \eta) = -\boldsymbol{W}(s, \eta)\sigma_{ss}(v^0; 0, s). \tag{2.13}$$

This fact, which follows, e.g., from the general results [4, Chapter 6], expresses the known Saint-Venant principle with allowance for conditions of periodicity on the sides of the half-band $\Pi(s)$. It should be noted that the decay of solution (2.13) of the boundary-layer type confirms the valid choice of the boundary conditions in problem (2.3), (2.4) for the regular-type solution.

It is clear that $\boldsymbol{W}$ is the solution of problem (2.11) with the right sides $\boldsymbol{F}_1 = \boldsymbol{F}_2 = 0$, $\boldsymbol{G}_1 = 0$, and $\boldsymbol{G}_2 = \boldsymbol{\nu}_2$. With the use of this solution, we introduce the energy characteristic of the elastic half-band, which is similar to the elastic capacity used in [5], by the equation

$$m(s) = \boldsymbol{E}(\boldsymbol{W}; \Pi(s)) = \frac{1}{2} \sum_{j,k=1}^{2} \int\limits_{\Pi(s)} \boldsymbol{\sigma}_{jk}(\boldsymbol{W}; s, \eta)\, \boldsymbol{\varepsilon}_{jk}(\boldsymbol{W}; s, \eta)\, d\eta. \tag{2.14}$$

The quantity (2.14) is nonnegative and degenerates only if $\pi(s)$ is a segment parallel to the axis η_2 and, hence, $\boldsymbol{W} = 0$.

3. Minor Terms of the Asymptotic. The second term v^1 of the regular type satisfies the equations

$$-\partial_{x_1} \sigma_{1k}(v^1; x) - \partial_{x_2} \sigma_{2k}(v^1; x) = \rho(x)(\lambda_0 v_k^1(x) + \lambda_1 v_k^0(x)), \qquad x \in \Omega, \tag{3.1}$$

and the right sides of the boundary conditions

$$\sigma_k^{(n)}(v^1; 0, s) = g_k^1(s), \qquad s \in \Gamma \tag{3.2}$$

are determined from the conditions of decay of the solution w^1 of the boundary-layer type satisfying problem (2.11) with $i = 1$. We find the right sides $\boldsymbol{F}^1$ and $\boldsymbol{G}^1$ of problem (2.11) contained in these conditions, as was mentioned in [4, Chapter 6]:

$$\int\limits_{\Pi(s)} \boldsymbol{F}_k^1(s, \eta)\, d\eta + \int\limits_{\pi(s)} \boldsymbol{G}_k^1(s, \eta)\, ds_\eta = 0, \qquad k = 1, 2. \tag{3.3}$$

The components of the vector function $\boldsymbol{F}^1$ acquire the form

$$\begin{aligned} \boldsymbol{F}^1 &= \partial_s \boldsymbol{\sigma}_{12}(\boldsymbol{w}^0) + \varkappa(\boldsymbol{\sigma}_{11}(\boldsymbol{w}^0) - \boldsymbol{\sigma}_{22}(\boldsymbol{w}^0)) - \varkappa \eta_1 \partial_{\eta_2} \boldsymbol{\sigma}_{12}(\boldsymbol{w}^0) + \partial_{\eta_1} \Sigma_{11} + \partial_{\eta_2} \Sigma_{12}, \\ \boldsymbol{F}^2 &= \partial_s \boldsymbol{\sigma}_{22}(\boldsymbol{w}^0) + 2\varkappa \boldsymbol{\sigma}_{12}(\boldsymbol{w}^0) - \varkappa \eta_1 \partial_{\eta_2} \boldsymbol{\sigma}_{22}(\boldsymbol{w}^0) + \partial_{\eta_1} \Sigma_{21} + \partial_{\eta_2} \Sigma_{22}. \end{aligned} \tag{3.4}$$

The first terms in the right sides of Eqs. (3.4) appear in accordance with the rule of differentiation of the complex function

$$\frac{dz}{ds}\Big(s, \frac{s}{h}\Big) = \Big(\frac{\partial z}{\partial s}(s, \eta_2) + \frac{1}{h}\, \frac{\partial z}{\partial \eta_2}(s, \eta_2)\Big)\Big|_{\eta_2 = h^{-1} s}, \tag{3.5}$$

the second and third terms appear because the equilibrium equations (2.9) contain the curvature $\varkappa(s)$ and the Jacobian $J(n, s)^{-1} = 1 - h\eta_1 \varkappa(s) + O(h^2)$, and the fourth and fifth terms appear because it is necessary to take into account relations (2.8), (3.5) and the Taylor expansion

$$A(n, s) = A(0, s) + h\eta_1 \partial_n A(0, s) + O(h^2) \tag{3.6}$$

in calculating the true stresses $\sigma_{pq}(w^0)$. The explicit form of the expressions Σ_{jk} is not needed, because they are canceled in formula (3.3) together with similar terms in the components of the vector function $\boldsymbol{G}^1$:

$$\begin{aligned} \boldsymbol{G}_1^1 &= -\boldsymbol{\nu}_1 \Sigma_{11} - \boldsymbol{\nu}_2 \Sigma_{12} - H(\boldsymbol{\nu}_1\, \partial_n \sigma_{nn}(v^0) + \boldsymbol{\nu}_2\, \partial_n \sigma_{sn}(v^0)) \\ &+ Y(\boldsymbol{\sigma}_1^{(\nu)}(\boldsymbol{w}^0) + \boldsymbol{\sigma}_1^{(\nu)}(\boldsymbol{v}^0)) + (\boldsymbol{\nu}_2 H \varkappa + y\, \partial_s H)(\boldsymbol{\sigma}_{12}(\boldsymbol{w}^0) + \sigma_{sn}(v^0)), \\ \boldsymbol{G}_2^1 &= -\boldsymbol{\nu}_1 \Sigma_{21} - \boldsymbol{\nu}_2 \Sigma_{22} - H(\boldsymbol{\nu}_1\, \partial_n \sigma_{ns}(v^0) + \boldsymbol{\nu}_2 \partial_n \sigma_{ss}(v^0)) \\ &+ Y(\boldsymbol{\sigma}_1^{(\nu)}(\boldsymbol{w}^0) + \boldsymbol{\sigma}_2^{(\nu)}(\boldsymbol{v}^0)) + (\boldsymbol{\nu}_2 H \varkappa + y\, \partial_s H)(\boldsymbol{\sigma}_{22}(\boldsymbol{w}^0) + \sigma_{ss}(v^0)). \end{aligned} \tag{3.7}$$

The terms with the factor H appear from the Taylor formula for stresses $\sigma(v^0; n, s)$ similar to Eq. (3.6), and the last pairs of terms in the right sides of equalities (3.7) originate from the second terms of expansions (2.10) of the components of the normal ν. All stresses generated by the displacement field v^0 are calculated on the contour, i.e., the boundary conditions (2.4) leave only the longitudinal stress $\sigma_{ss}(v^0; 0, s)$ with a nonzero value. The multipliers at Y are equal to zero by virtue of conditions on the butt-end face $\pi(s)$ in problem (2.11) for $\boldsymbol{w}^0$. The derivatives $\partial_n \sigma_{nn}(v^0; 0, s)$ and $\partial_n \sigma_{ns}(v^0; 0, s)$ can be found from the equilibrium equations (2.9). Finally, the Stokes formula

$$\int\limits_{\Pi(s)} \eta_1\, \frac{\partial}{\partial \eta_2}\, \boldsymbol{\sigma}_{j2}(\boldsymbol{w}^0; s, \eta)\, d\eta = -\int\limits_{\pi(s)} \eta_1 \boldsymbol{\nu}_2(s, \eta_2) \boldsymbol{\sigma}_{j2}(\boldsymbol{w}^0; s, \eta)\, ds_\eta$$

shows that several terms in Eqs. (3.4) and (3.7), which contain the products of the integrands and the curvature $\varkappa$, do not contribute to conditions (3.3).

To find the integrals of the remaining terms in Eqs. (3.4) and (3.7), we use several formulas:

1) relations

$$\int\limits_{\pi(s)} \boldsymbol{\nu}_j(s,\eta_2)\,ds_\eta = \delta_{j,1}, \qquad \int\limits_{\pi(s)} \eta_1 \boldsymbol{\nu}_j(s,\eta_2)\,ds_\eta = \overline{H}(s)\delta_{j,1},$$

$$\int\limits_{\pi(s)} y(s,\eta)\,\frac{\partial H}{\partial s}(s,\eta_2)\,ds_\eta = \int\limits_0^1 \frac{\partial H}{\partial s}(s,\eta_2)\,d\eta_2 = \frac{d\overline{H}}{ds}(s), \qquad \overline{H}(s) = \int\limits_0^1 H(s,\eta_2)\,d\eta_2, \tag{3.8}$$

where $\delta_{j,k}$ is the Kronecker delta;

2) equalities

$$\int\limits_{\pi(s)} \boldsymbol{\nu}_2(s,\eta)\boldsymbol{W}_2(s,\eta)\,ds_\eta = \sum_{j=1}^{2} \int\limits_{\pi(s)} \boldsymbol{\sigma}_j^{(\nu)}(\boldsymbol{W};s,\eta)\boldsymbol{W}_j(s,\eta)\,ds_\eta = 2m(s),$$

$$\sum_{j,k=1}^{2} \boldsymbol{A}_{jk}^{1i}(0,s) \int\limits_{\pi(s)} \boldsymbol{\nu}_k(s,\eta_2)\boldsymbol{W}_j(s,\eta_2)\,ds_\eta = 0 \qquad (i=1,2); \tag{3.9}$$

3) equality

$$\varepsilon_{ss}(v^0;0,s) = b(0,s)\sigma_{ss}(v^0;0,s) \tag{3.10}$$

valid by virtue of the boundary conditions (2.4) and containing the element $b(0,s) = \boldsymbol{B}_{22}^{22}(0,s)$ of the compliance tensor $\boldsymbol{B}(0,s)$ inverse to the stiffness tensor $\boldsymbol{A}(0,s)$. For a homogeneous isotropic material, we have $b(0,s) = (2\mu)^{-1}(1-\nu)$. Here μ is the shear modulus and ν is Poisson's ratio. To check the last two equalities in (3.9), we apply the Green formula with the fields $\boldsymbol{W}(s,\eta)$ and $\boldsymbol{\zeta}^1(\eta) = (\eta_1, 0)$ or $\boldsymbol{\zeta}^2(\eta) = (0,\eta_1)$:

$$0 = \sum_{j,k=1}^{2} \int\limits_{\Pi(s)} \boldsymbol{\zeta}_j^i(\eta)\,\frac{\partial}{\partial \eta_k}\,\boldsymbol{\sigma}_{jk}(\boldsymbol{W};s,\eta)\,d\eta = \sum_{j,k=1}^{2} \int\limits_{\pi(s)} \Big(\boldsymbol{W}(s,\eta)\boldsymbol{\sigma}^{(\nu)}(\boldsymbol{\zeta}^i;\eta) - \boldsymbol{\zeta}_k^i(\eta)\boldsymbol{\sigma}_k^{(\nu)}(\boldsymbol{W};s,\eta)\Big)\,ds_\eta.$$

The expression in the right side of this formula coincides with expression (3.9) with $i = 1,2$ by virtue of relations (3.8), (2.12), and (2.13), and also equalities (2.7) for the vector polynomials $\boldsymbol{\zeta}^i(\eta)$.

Finally, we obtain the formulas

$$\int\limits_{\Pi(s)} \frac{\partial}{\partial s}\,\boldsymbol{\sigma}_{p2}(\boldsymbol{W};s,\eta)\,d\eta - \int\limits_{\pi(s)} y(s,\eta)\,\frac{\partial H}{\partial s}(s,\eta)\,\boldsymbol{\sigma}_{p2}(\boldsymbol{W};s,\eta)\,ds_\eta = \frac{\partial}{\partial s} \int\limits_{\Pi(s)} \boldsymbol{\sigma}_{p2}(\boldsymbol{W};s,\eta)\,d\eta,$$

$$\int\limits_{\Pi(s)} \boldsymbol{\sigma}_{1i}(\boldsymbol{W};s,\eta)\,d\eta = \sum_{j,k=1}^{2} \boldsymbol{A}_{jk}^{1i}(0,s) \int\limits_{\Pi(s)} \boldsymbol{\varepsilon}_{jk}(\boldsymbol{W};s,\eta)\,d\eta = \sum_{j,k=1}^{2} \boldsymbol{A}_{jk}^{1i}(0,s) \int\limits_{\pi(s)} \boldsymbol{\nu}_k(s,\eta_2)\boldsymbol{W}_j(s,\eta)\,ds_\eta = 0, \tag{3.11}$$

$$\int\limits_{\Pi(s)} \boldsymbol{\sigma}_{22}(\boldsymbol{W};s,\eta)\,d\eta = b(0,s)^{-1} \int\limits_{\Pi(s)} \boldsymbol{\varepsilon}_{22}(\boldsymbol{W};s,\eta)\,d\eta - \sum_{i=1}^{2} \beta_i(s) \int\limits_{\Pi(s)} \boldsymbol{\sigma}_{1i}(\boldsymbol{W};s,\eta)\,d\eta$$

$$= b(0,s)^{-1} \int\limits_{\pi(s)} \boldsymbol{\nu}_2(s,\eta)\boldsymbol{W}_2(s,\eta)\,ds_\eta = 2b(0,s)^{-1}m(s).$$

In the last calculations, the stress $\boldsymbol{\sigma}_{22}$ is expressed via the strain $\boldsymbol{\varepsilon}_{22}$ and the stresses $\boldsymbol{\sigma}_{11}$ and $\boldsymbol{\sigma}_{12}$ with certain coefficients β_1 and β_2 [cf. Eq. (3.10)], and also equalities (3.9). Relation (3.11) involves the formula of differentiation of the integral with a variable upper limit; according to definition (2.10) of the factor $y(s,\eta)$, we have $y(s,\eta)\,ds_\eta = d\eta_2$.

Cumbersome though simple calculations based on the formulas derived above yield the following expressions for the right sides of the boundary conditions (3.2):

$$\begin{aligned} g_n^1(s) &= -\varkappa(s)\alpha(s)\varepsilon_{ss}(v^0;0,s) + \lambda_0\overline{H}(s)\rho(0,s)v_n^0(0,s), \\ g_s^1(s) &= \partial_s(\alpha(s)\varepsilon_{ss}(v^0;0,s)) + \lambda_0\overline{H}(s)\rho(0,s)v_s^0(0,s); \end{aligned} \tag{3.12}$$

$$\alpha(s) = b(0,s)^{-1}\overline{H}(s) - 2b(0,s)^{-2}m(s). \tag{3.13}$$

4. Comments on Asymptotics Appropriateness. Let $\lambda_0 = \lambda^{(n)}$ be an element of a sequence of eigenvalues of the limiting problem (2.3), (2.4):

$$0 = \lambda^{(1)} = \lambda^{(2)} = \lambda^{(3)} < \lambda^{(4)} \le \lambda^{(5)} \le \ldots \le \lambda^{(n)} \le \ldots \to +\infty. \tag{4.1}$$

The zero eigenvalue corresponds to stiff displacements. We make an assumption on the multiplicity of the eigenvalue $\lambda^{(n)} = \varkappa \ge 1$, i.e.,

$$\lambda^{(n-1)} < \lambda^{(n)} = \ldots = \lambda^{(n+\varkappa-1)} < \lambda^{(n+\varkappa)},$$

and use $v^{(n)}, \ldots, v^{(n+\varkappa-1)}$ to denote the corresponding vector functions subjected to conditions of orthogonality and normalization:

$$\sum_{j=1}^{2} \int\limits_{\Omega} \rho(x)v_j^{(p)}(x)v_j^{(q)}(x)\,dx = \delta_{p,q}. \tag{4.2}$$

We use the following terms as the initial terms of ansatzes (2.1) and (2.2):

$$\lambda_0 = \lambda^{(n)}, \qquad v^0 = a_1 v^{(n)} + \ldots + a_\varkappa V^{(n+\varkappa-1)}.$$

We find the number λ_1 and the column $a = (a_1, \ldots, a_\varkappa)$ from the conditions of solvability of problem (3.1), (3.2) for the field v^1, which acquire the following form by virtue of relations (3.12), (2.8), and (4.2):

$$\begin{gathered} a_p\lambda_1 = -\sum_{j=1}^{2} \int\limits_{\Gamma} g_j^1(s)v^{(n+p)}(0,s)\,ds = \sum_{p=0}^{\varkappa-1} M_{pq}^{(n)} a_q, \\ M_{pq}^{(n)} = \int\limits_{\Gamma} \Big(\alpha(s)\varepsilon_{ss}(v^{(n+p)};0,s)\varepsilon_{ss}(v^{(n+q)};0,s) - \lambda^{(n)}\rho(0,s)\overline{H}(s)\sum_{j=1}^{2} v_j^{(n+p)}(0,s)v_j^{(n+q)}(0,s)\Big)\,ds_x. \end{gathered} \tag{4.3}$$

Clearly, the matrix $M^{(n)}$ with elements (4.3) is symmetric, i.e., it has real numbers $\lambda_1^{n0}, \ldots, \lambda_1^{(n\varkappa-1)}$ and its own orthonormalized columns $a^{n0}, \ldots, a^{(n\varkappa-1)}$, which define concretely the terms λ_0, λ_1 and v^0, w^0 of ansatzes (2.1) and (2.2); moreover, from problems (3.1), (3.2), and (2.11), where $i = 1$, one can find the next asymptotic terms $v^1(x)$ and $w^1(s,\eta)$ (note that they are determined with accuracy to the terms $\tilde{v}(x) = \tilde{a}_0 v^{(n)}(x) + \ldots + \tilde{a}_\varkappa v^{(n+\varkappa-1)}(x)$ and $-\boldsymbol{W}(s,\eta)\sigma_{ss}(\tilde{v};0,s)$).

Despite oscillations of the boundary (1.1) of the domain Ω_h, Korn's inequality is valid [6, Chapter 3]:

$$\|u; H^1(\Omega_h)\|^2 \le c \sum_{j,k=1}^{2} \|\varepsilon_{jk}(u); L_2(\Omega_h)\|^2; \tag{4.4}$$

in this inequality, the constant c is independent of the parameter $h \in (0, h_0]$, $h_0 > 0$ and the vector function u belonging to the Sobolev space $H^1(\Omega_h)$ and subjected to the orthogonality conditions

$$\int\limits_{\omega} u_1(x)\,dx = \int\limits_{\omega} u_2(x)\,dx = \int\limits_{\omega} (x_2u_1(x) - x_1u_2(x))\,dx = 0,$$

where ω is a nonempty subdomain in Ω, for instance, $\omega = \Omega \setminus \overline{O_\Gamma}$. Inequality (4.4) offers a partial substantiation of the asymptotics constructed by means of a standard approach (see, e.g., [7]), namely, by means of a lemma "on almost eigenvalues and vectors" [8], one can easily verify that the sequence of eigenvalues of problem (1.2), (1.3) similar to (4.1)

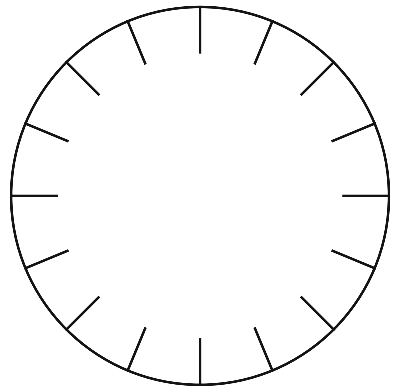

Fig. 2. Body with a periodic family of edge cracks.

$$0 = \Lambda^{(1)} = \Lambda^{(2)} = \Lambda^{(3)} < \Lambda^{(4)} \le \Lambda^{(5)} \ldots \le \Lambda^{(n)} \le \ldots \to +\infty \tag{4.5}$$

contains at least $\varkappa$ elements $\Lambda^{(p)}, \ldots, \Lambda^{(p+\varkappa-1)}$ for which the following estimates are valid:

$$|\Lambda^{(p)} - \lambda^{(n)} - h\lambda_1^{(nq)}| \le c_{nq} h^{3/2} \qquad (q = 0, \ldots, \varkappa - 1). \tag{4.6}$$

It is more difficult to check the following conclusions: there are no more than $\varkappa$ of these elements and the equality $p = n$ is valid. Usually these conclusions are verified on the basis of the "convergence" theorem, which predicts that $\Lambda_q(h) \to \lambda_q$ as $h \to +0$ (see [9, 7] and other publications). In [6] (see also [9–12]), procedures of direct and inverse convergence are proposed, which, first, establish the two facts mentioned above and, second, reveal the dependence of the multipliers c_{nq} in relations (4.6) on the characteristics of the limiting spectrum: eigenvalue $\lambda^{(n)}$, its multiplicity $\varkappa = \varkappa^{(n)}$, and relative distance $d_n = \min\{1 - \lambda^{(n-1)}/\lambda^{(n)}, 1 - \lambda^{(n)}/\lambda^{(n+\varkappa)}\}$ to the neighboring points of the spectrum. The mere formulation of the latter result is rather cumbersome; therefore, we restrict ourselves to statement of a typical conclusion verified by the known scheme: inequalities (4.6) contain the eigenvalues $\Lambda^{(n)}, \ldots, \Lambda^{(n+\varkappa-1)}$, and the remaining terms of sequence (4.5) do not satisfy these inequalities.

The results obtained also remain valid for more generic types of boundary perturbations. As an example, we can consider a periodic family of edge cracks (Fig. 2). In this case, $\overline{H}(s) = 0$ and $m(s) > 0$; hence, coefficient (3.13) is negative.

5. Modeling with the Help of Wenzel's Conditions. The publications dealing with investigations of the boundary layer in the theory of thin elastic plates [13, 14] and with singularly perturbed scalar spectral boundary-value problems [1] offer a method for unifying the limiting problems of the form (2.3), (2.4) and (3.1), (3.2) into a general (so-called resulting) boundary-value problem whose solution is an elevated-accuracy approximation, as compared with the solution of the original problem (1.2), (1.3). In the case considered, such a problem has the form

$$\begin{gathered}
-\partial_{x_1}\sigma_{1k}(u^w; h, x) - \partial_{x_2}\sigma_{2k}(u^w; h, x) = \Lambda^w(h)\rho(x)u_k^w(h, x), \qquad x \in \Omega, \\
\sigma_{nn}(u^w; h, 0, s) + h\varkappa(s)\varepsilon_{ss}(u^w; h, 0, s) = \Lambda^w(h)\rho(0, s)\overline{H}(s)u_n^w(h, 0, s), \\
\sigma_{ns}(u^w; h, 0, s) - h\,\partial_s(\alpha(s)\varepsilon_{ss}(u^w; h, 0, s)) = \Lambda^w(h)\rho(0, s)\overline{H}(s)u_s^w(h, 0, s), \qquad s \in \Gamma.
\end{gathered} \tag{5.1}$$

As the last boundary condition contains a second-order differential operator $-h\alpha(s)\,\partial_s^2$, the boundary conditions should be interpreted as an elastic analog of Wenzel's scalar condition (see [15] and the references therein). In [16–18], similar boundary conditions were called the wall-law conditions.

Problem (5.1) has the following variational formulation [19, 20]: it is necessary to find a number $\Lambda^w = \Lambda^w(h)$ and a nontrivial vector function u^w that belongs to the space $H^1(\Omega)$, has a trace at the boundary Γ from the Sobolev space $H^1(\Gamma)$, and satisfies the following integral identity for all test functions v^w with similar properties:

$$2E(u^w, v^w; \Omega) + 2S(u^w, v^w; \Gamma) = \Lambda^w \sum_{j=1}^{2} \Big(\int_\Omega \rho u_j^w v_j^w \, dx + \int_\Gamma \overline{H}\rho u_j^w v_j^w \, ds_x \Big). \tag{5.2}$$

Here E and S are the elastic and surface energies:

$$E(u^w, v^w; \Omega) = \frac{1}{2} \sum_{j,k=1}^{2} \int_{\Omega} \sigma_{jk}(u^w) \varepsilon_{jk}(u^w)\, dx, \qquad S(u^w, v^w; \Gamma) = \frac{h}{2} \int_{\Gamma} \alpha \varepsilon_{ss}(u^w) \varepsilon_{ss}(v^w)\, ds_x.$$

If the conditions

$$\alpha(s) > 0, \qquad \overline{H}(s) \geq 0 \tag{5.3}$$

are satisfied [see Eqs. (3.13) and (3.8)], the left side of identity (5.2) determines the scalar product in the above-described energy class, and the multiplier at Λ^w in the right side is a positive compact operator. Hence, problem (5.2) or, which is the same, problem (5.1) has a sequence of eigenvalues $\Lambda^w(n)$ of the form (4.5).

If the inequality $\overline{H}(s) < 0$ is satisfied on a segment of the contour Γ of positive length, then the eigenvalues $\{\Lambda^w(n)\}$ include an infinite set of negative numbers. If the function α is negative (see the comments to Fig. 2 at the end of Sec. 4), only several eigenvalues are nonnegative, while the other eigenvalues form a large negative sequence. Finally, if the function α degenerates at a point or on a segment of the contour Γ, the spectrum of problem (5.1) becomes continuous. In the situations described above, problem (5.1) cannot serve as a model for problem (1.2), (1.3), i.e., inequalities (5.3) are conditions necessary during modeling.

The second restriction (5.3) is purely geometric, while the first one, according to presentation (3.13), contains both the geometric and the energy characteristics of the rapidly oscillating boundary Γ_h. Moreover, it is only in an obviously unacceptable case $\overline{H}(s) < 0$ that the value of $\alpha(s)$ is definitely negative, which is also unacceptable. If $\overline{H}(s) \geq 0$, then the value of $\alpha(s)$ may be either positive or negative. As the substitution

$$H(s, \eta_2) \mapsto H(s, \eta_2) + H_0 \tag{5.4}$$

does not affect the energy characteristic (2.14), we may conclude that both conditions (5.3) are satisfied for a new contour $\{x \in O_\Gamma\colon\ n = hH(s, h^{-1}s) + hH_0\}$ if H_0 in Eq. (5.4) has a sufficiently large positive constant value.

The procedure of constructing the asymptotics of the solutions of problem (5.1) is fairly simple: the boundary layer disappears from the corresponding ansatzes

$$\Lambda^w(h) = \lambda_0 + h\lambda_1 + \dots\ , \qquad u^w(h, x) = v^0(x) + hv^1(x) + \dots\ , \tag{5.5}$$

and their terms satisfy problems (2.3), (2.4) and (3.1), (3.2). Justification of the asymptotic expansions (5.5) as a whole follows the scheme described in Sec. 4 (see also [1]). The similarity of formulas (2.1), (2.2), and (5.5) suggests that problem (5.1) with Wenzel's boundary conditions under restrictions (5.3) is a model of problem (1.2), (1.3) in the domain Ω_h with a rapidly oscillating boundary. Relations (4.6) for the eigenvalues $\Lambda^{(n)}(h)$ and $\Lambda^{w(n)}(h)$ yield the inequalities

$$|\Lambda^{(n)}(h) - \Lambda^{w(n)}(h)| \leq C_n^w h^{3/2}. \tag{5.6}$$

A comparison of the eigen vector functions $u^{(n)}$ and $u^w(n)$ may be performed in the L_2 metric, because the energy norm of the term $h\chi w^1$ from ansatz (2.2) is $O(h^{1/2})$, i.e., the boundary layer prevails over the correction hv^1 of the regular type.

6. Smooth Image of a Rapidly Oscillating Boundary. Another approach, which differs from that described in Sec. 5, was proposed in [1]. This approach is based on the following observation: if we take a quantity $H^s(s)$ independent of the fast variable instead of $H(s, h^{-1}s)$ in Eq. (1.1), all the results described above remain valid, because a constant function is a particular case of a periodic function. At the same time, we have $\boldsymbol{W} = 0$, $m = 0$ and $\overline{H^s}(s) = H^s(s)$, $\alpha^s(s) = b(0, s)^{-1} H^s(s)$ in relations (2.13), (2.14) and (3.13), (3.12).

We assume that

$$H^s(s) = \overline{H}(s) - 2b(0, s)^{-1} m(s), \tag{6.1}$$

introduce a regularly perturbed domain Ω_h^s bounded by the contour $\Gamma_h^s = \{x \in O_\Gamma\colon\ n = hH^s(s)\}$, and consider the spectral problem

$$-\partial_{x_1} \sigma_{1k}(u^s; h, x) - \partial_{x_1} \sigma_{2k}(u^s; h, x) = \Lambda^s(h) \rho(x) u_k^s(h, x), \qquad x \in \Omega_h^s; \tag{6.2}$$

$$\sigma_k^{(n^s)}(u^s; h, x) = \Lambda^s(h) h (\overline{H}(s) - H^s(s)) \rho(0, s) u_k^s(h, x), \qquad x \in \Gamma_h^s, \tag{6.3}$$

where n^s is the unit vector of the external normal to the boundary $\partial\Omega_h^s = \Gamma_h^s$. For problem (6.2) to be solved, we retain the asymptotic ansatzes (5.5) where the superscript w is replaced by s. As the right side of the boundary condition (6.3) includes the expression with the small parameter h and by virtue of Eq. (6.1), the terms of these ansatzes satisfy the same problems (2.3), (2.4) and (3.1), (3.2) as the terms of ansatzes (2.1) and (2.2). Thus, problem (6.2), (6.3) is a model with higher accuracy for problem (1.2), (1.3) in a domain with a rapidly oscillating boundary. As the domain Ω_h^s is perturbed in a regular manner, justification of the asymptotics is provided by the general results of functional analysis (see, e.g., [21]).

Let us demonstrate the variational formulation of problem (6.2), (6.3). We have to find a number Λ^s and a nontrivial vector function $u^s \in H^1(\Omega_h^s)$ for which the following integral identity is satisfied for all test functions $v^s \in H^1(\Omega_h^s)$:

$$2E(u^s, v^s; \Omega_h^s) = \Lambda^s \sum_{j=1}^{2} \Big(\int\limits_{\Omega_h^s} \rho u_j^s v_j^s \, dx + \int\limits_{\Gamma_h^s} h(\overline{H} - H^s) \rho u_j^s v_j^s \, ds_x \Big). \tag{6.4}$$

As $H^s \geq \overline{H}$ by virtue of relations (6.1) and (2.14), the multiplier at Λ^s in identity (6.4) determines a positive compact operator in the space $H^1(\Omega_h^s)$; hence, the variational problem (6.4) and the boundary-value problem (6.2), (6.3) have a sequence of eigenvalues $\Lambda^s(n)$ of the form (4.5). Moreover, applying inequalities (4.6) to two problems (in the domain Ω_h and in the domain Ω_h^s), we can demonstrate that their eigenvalues are related by Eq. (5.6) with the subscript w replaced by s.

From the viewpoint of calculations, solving problem (6.2), (6.3) is much simpler than solving problems (1.2), (1.3), and (5.1) with a rapidly oscillating boundary and a small parameter at higher derivatives, respectively. Therefore, the principle of a smooth image of a singularly perturbed boundary proposed in [1] is more convenient for modeling than Wenzel's near-wall boundary conditions. It is impossible, however, to eliminate the spectral parameter from the boundary condition (6.3): the quantity $b(0, s)(\overline{H}(s) - H^s(s))/2 \geq 0$ is the energy characteristic (2.14) of oscillations of contour (1.1), which vanishes only in the case of a regular perturbation of the body, i.e., in the absence of oscillations of the body surface. This observation again confirms that the model of a fine-grain boundary cannot be constructed on the basis of geometric measurements only.

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